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Reformulation of Edelbaum's Low-Thrust Transfer Problem Using Optimal Control Theory

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The problem of optimal low-thrust transfer between inclined orbits is reformulated within the framework of optimal control theory. The original treatment considered the time-constrained inclination maximization with velocity as the independent variable allowing the use of the theory of maxima. Because the independent variable is double valued for some transfers, two expressions for the inclination change involving inverse-sine functions are needed to describe all possible transfers. The present analysis casts this problem as a minimum-time transfer between given noncoplanar circular orbits and obtains a single analytic expression for the orbital inclination involving a single inverse-tangent function, uniformly valid for all transfers. The ΔV penalty with respect to the exact transfer solution using the full six-state dynamic equations with optimized thrust profile during the transfer is shown to be small.

I. Introduction

ANALYTIC solutions of the low-thrust transfer problem are very useful in preliminary mission analysis as well as spacecraft systems design and optimization. The overall design of a solar-electric transfer vehicle or even an integrated spacecraft requires extensive parametric analyses for optimum sizing of the various power, propulsion, and thermal management systems to maximize delivered payload to the destination orbit. These parametric studies require hundreds of iterations, precluding the use of the numerically generated transfer solutions. The analytic solutions are also desirable for future onboard autonomous guidance applications, especially for smaller spacecraft such as in the mini- and microsatellite category where the application of low-thrust technology for orbit maintenance and control is most efficient.

In the early 1960s, Edelbaum[1,2] derived analytic expressions for the maximum change in inclination between two circular orbits of given size with continuous constant acceleration and fixed transfer time. Conversely, he derived an analytic expression for the total ΔV needed to carry out the transfer between given inclined circular orbits. This theory was later generalized by Wiesel and Alfano,[3] who allowed for the variation of the out-of-plane or thrust yaw angle during each revolution, unlike Edelbaum, who used the simpler constant yaw profile. Thus, the (a, i) semimajor axis and inclination space was mapped by direct numerical integration of the simplified differential equations in a and i, such that the minimum time for a given transfer is read directly from the solution map.

In Refs. 4 and 5, the optimal thrust pitch and yaw profiles required for a given transfer were determined in a semianalytic way by also considering discontinuous thrust due to eclipsing. However, these solutions are not analytic and, therefore, are difficult to implement in systems design optimization software. In Ref. 6, rapid transfer calculations were demonstrated by analytic modeling of the various transfer parameters including shadowing and solar power degradation effects due to the Van Allen radiation belts. The thrust yaw angle is held constant throughout the transfer, and the required value is determined by iteration. This is not as optimal as the Edelbaum steering solution, which holds the yaw angle constant during each revolution but varies its value from revolution to revolution in an optimal manner. All of these analyses assume that the orbit remains or is forced to be circular after each cycle or revolution.

In Refs. 7 and 8, piecewise constant steering angles in both pitch and yaw are solved for, to effect the largest change in semimajor axis and inclination, respectively, in the presence of a given shadow arc where no thrust is applied. These piecewise constant thrust profiles can be easily implemented in software, such as depicted in Ref. 6. These discussions show the need to develop suboptimal solutions that are as realistic as possible in including the various physical, geometric, and other effects but that are analytic for the reasons just mentioned.

It is within this context that the original Edelbaum theory is revisited by first extending and completing it and by deriving the yaw steering expressions without ambiguity and later recasting the theory within the framework of optimal control. Finally, an algorithm is developed for direct implementation in mission analysis and parametric studies software. Assuming constant acceleration and constant thrust vector yaw angle within each revolution, Edelbaum linearized the Lagrange planetary equations of orbital motion about a circular orbit and, using the velocity as the independent variable, he reduced the transfer optimization problem to a problem in the theory of maxima. The variational integral involved a single constant Lagrange multiplier because it involved a single integral constraint equation for the transfer time or velocity change, while maximizing the change in orbital inclination. The control variable being the yaw angle, the necessary condition for a stationary solution was obtained by simply setting the partial derivative of the integrand of the variational integral with respect to the control to zero. This optimum control was then used in the right-hand sides of the original equations of motion, which were integrated analytically to provide expressions for the time and inclination in terms of the independent variable, the orbital velocity. However, the analytic expression for the inclination is given in terms of a certain inverse-sine function, which is a double-valued function in the interval $(0, 2\pi)$, for a given value of its argument, which includes the velocity. It is for this reason that two expressions for the inclination must be provided to cover the case of large inclination change transfers. For these transfers, it is necessary to use one of these expressions first to describe the transfer up to a certain point in time and switch to the other expression for the balance of the transfer. A certain condition involving current time, initial velocity, and initial yaw angle must be monitored to evaluate the time at which the switch from one expression to the other must take place for a continuous description of the overall inclination change history. This complication was introduced as a result of the formulation used by Edelbaum, who adopted orbital velocity as the independent variable, which led to the integration of the differential equation for the inclination with respect to the velocity.

This unnecessary difficulty is removed, and a single expression for the inclination change, uniformly valid throughout any desired transfer, is obtained by the present analysis, which also casts the

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original Edelbaum problem into a minimum-time problem using the formalism of optimal control theory. In this analysis, inclination and velocity are the dependent variables, and time is the independent variable. The original linearized equations of orbital motion are used as the constraining differential equations to form the variational Hamiltonian involving the inclination and velocity adjoint variables. The inclination adjoint is determined to be constant, and the velocity adjoint is given simply in terms of the cosine of the single control variable, the yaw angle. All of the Edelbaum formulas are recovered, including the ΔV formula, which relates the total velocity change to the initial and final velocities as well as the total inclination change. This formula is valid for the range of total inclination changes $0 < \Delta i < 114.6$ deg because, for $\Delta i > 114.6$ deg, $\Delta V = V_0 + V_f$, the sum of the boundary velocities regardless of the magnitude of the inclination change. In the latter case, the transfer expands the orbit radius to infinity with cost V_0 where the rotation is carried out at no cost, and then it shrinks the radius to the final condition with cost V_f such that the total velocity change to be imparted to the transferring vehicle is the sum of the initial and final orbital velocities. The expressions for all of the pertinent state and control variables developed are simple and straightforward to use, and a simple algorithm shows how to apply them equally well for both the problems of transfer to higher orbit or lower orbit, irrespective of whether the initial inclination is smaller or larger than the final inclination. For autonomous onboard use, an electric orbit transfer vehicle can implement this analytic theory even though shadowing can somewhat complicate the transfer.

In Sec. II, Edelbaum's analysis is presented with many extensions and simplifications. Section III presents the formulation of Edelbaum's problem using optimal control theory with several transfer examples shown in Sec. IV, including a numerical comparison with the exact transfer solution.

II. Edelbaum's Original Analysis with Extensions

In this section, the Lagrange planetary equations linearized about a circular orbit are transformed further by averaging out the spacecraft position, and the system is reduced to two equations in i and orbit velocity V or a, inasmuch as the orbit is assumed to remain circular during the transfer. The velocity V is then made an independent variable at the expense of time t, which is now a dependent variable along i. The out-of-plane or thrust yaw angle β is obtained by requiring that an augmented integral remain stationary. This optimal solution is then used in the system differential equations, which, after analytic integration, provide expressions for the inclination and velocity describing their evolution during the transfer. We then extend Edelbaum's analysis and obtain unambiguous expressions for the steering law within the framework of his theory. The limitations and pitfalls in using Edelbaum's results are further explained, and the case is made for a more streamlined analysis using the techniques of optimal control theory with time as independent variable.

The full set of the Gaussian form of the Lagrange planetary equations for near-circular orbits is given by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2af_t}{V} \tag{1}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = \frac{2f_t c_\alpha}{V} - \frac{f_n s_\alpha}{V} \tag{2}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}t} = \frac{2f_t s_\alpha}{V} + \frac{f_n c_\alpha}{V} \tag{3}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{f_h c_\alpha}{V} \tag{4}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{f_h s_\alpha}{V s_i} \tag{5}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = n + \frac{2f_n}{V} - \frac{f_h s_\alpha}{V \tan i} \tag{6}$$

where s_α and c_α are $\sin\alpha$ and $\cos\alpha$, respectively; a is the orbit semimajor axis; i is the inclination; Ω is the right ascension of the ascending node; and $e_x = e\cos\omega$, $e_y = e\sin\omega$, with e and ω the orbital eccentricity and argument of perigee. Equations (2) and (3) replace the corresponding equations for $\dot{e}$ and $\dot{\omega}$ by using the nonsingular elements e_x and e_y because ω is poorly defined when e approaches zero. Finally, $\alpha = \omega + M$ represents the mean angular position, M the mean anomaly, and $n = (\mu/a^3)^{1/2}$ the orbit mean motion, with μ the Earth gravity constant. For near-circular orbits, we have $V = na = (\mu/a)^{1/2}$. The components of the thrust acceleration vector along the tangent, normal, and out-of-plane directions are given by f_t, f_n, and f_h with the normal direction oriented toward the center of attraction. Assuming only tangential and out-of-plane thrust acceleration, and therefore using $f_n = 0$, and assuming the orbit remains circular during the transfer, the preceding equations reduce to a set of four differential equations in a, i, Ω, and α. If f represents the magnitude of the acceleration vector and β the thrust yaw angle, then $f_t = fc_\beta$ and $f_h = fs_\beta$. In general, the thrust pitch angle is defined as the angle between the projection of the thrust vector onto the instantaneous orbit plane and the local horizontal direction, whereas the thrust yaw angle is defined as the angle between the thrust vector and its projection onto the instantaneous orbit plane. An alternate definition of the yaw angle consists of the angle between the projection of the thrust vector onto the local horizontal plane and the local horizontal direction. In this analysis, the pitch angle is set to zero, and the yaw angle complies with either of the two preceding definitions because in this case they are identical. Furthermore, $\alpha = \omega + M = \omega + \theta^* = \theta$, the angular position when $e = 0$, with $\theta = nt$ and θ^* the true anomaly. If the angle β is held piecewise constant, switching sign at the orbital antinodes, then the $f_h s_\alpha$ terms will have a net zero contribution such that the system of differential equations further reduces to

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2af_t}{V} \tag{7}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{c_\theta f_h}{V} \tag{8}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = n \tag{9}$$

We can now average out the angular position θ in Eq. (8) by integrating with respect to θ and by holding f, β, and V constant such as the averaged inclination rate is obtained:

$$\int_0^{2\pi} \left(\frac{\mathrm{d}i}{\mathrm{d}t}\right) \mathrm{d}\theta = \frac{2fs_\beta}{V} \int_{-\pi/2}^{\pi/2} c_\theta \,\mathrm{d}\theta \qquad \frac{\mathrm{d}i}{\mathrm{d}t} = \frac{2fs_\beta}{\pi V} \tag{10}$$

Because of averaging, β is now a continuous function of time. From the energy equation $V^2/2 - \mu/r = -\mu/2a$, with $r = a$, and using Eq. (7) to eliminate the semimajor axis, the time rate of change of the velocity is obtained as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -fc_\beta \tag{11}$$

Equations (10) and (11) can be replaced by the following set, where V is now the independent variable and i and t are the dependent variables:

$$\frac{\mathrm{d}i}{\mathrm{d}V} = -\frac{2\tan\beta}{\pi V} \tag{12}$$

$$\frac{\mathrm{d}t}{\mathrm{d}V} = -\frac{1}{fc_\beta} \tag{13}$$

Let I represent the functional to be maximized,

$$I = \int_{V_0}^{V_f} \left(\frac{\mathrm{d}i}{\mathrm{d}V}\right) \mathrm{d}V = -\int_{V_0}^{V_f} \frac{2}{\pi V} \tan\beta \,\mathrm{d}V$$

and let J represent the integral constraint, with the constant being equal to t_f, the total transfer time,

$$J = \int_{V_0}^{V_f} \left(\frac{\mathrm{d}t}{\mathrm{d}V}\right) \mathrm{d}V = \text{const}$$

Edelbaum adjoins J to I by way of a constant Lagrange multiplier λ such that the optimization problem is now reduced to a succession of ordinary maximum problems for each value of V between V_0 and V_f, the initial and final velocities, respectively. The necessary condition for a stationary solution of the augmented integral K

$$K = I + \lambda J = \int_{V_0}^{V_f} \left[-\frac{2}{\pi V} \tan\beta - \frac{\lambda}{f c_\beta} \right] \mathrm{d}V$$

(Refs. 2, 9, and 10) is then simply given by

$$\frac{\partial}{\partial\beta} \left[\frac{2}{\pi V} \tan\beta + \frac{\lambda}{f c_\beta} \right] = 0$$

The optimization problem consists, therefore, of the maximization of the inclination change subject to the constraint of given total transfer time because, with $t = t_f$, $\Delta V = f \cdot t = f \cdot t_f$. This constraint is equivalent to the fixed ΔV constraint for constant acceleration f. Furthermore, V_0 and V_f being given, the initial and final radii, therefore, are given, too, because the orbits are assumed circular. The acceleration being applied continuously, this problem is equivalent to minimizing the total transfer time for a given change in the inclination and velocity. This is also equivalent to minimizing the total ΔV or propellant usage because the thrust is always on and no coasting arcs are allowed. In the latter case, I and J are simply interchanged to yield the optimality condition

$$\frac{\partial}{\partial\beta} \left[\frac{1}{f c_\beta} + \lambda_i \frac{2}{\pi V} \tan\beta \right] = 0$$

From the necessary condition, it follows that

$$V s_\beta = -(2f/\pi\lambda) = \text{const} = V_0 s_{\beta_0} \tag{14}$$

$$\lambda = -\frac{2f}{\pi V_0 s_{\beta_0}} \tag{15}$$

The optimal β steering law given by Eq. (14) can be used in Eq. (13) for $\mathrm{d}V/\mathrm{d}t$ to obtain the expression for the velocity as a function of time t or ΔV because f is constant and $\Delta V = f \cdot t$:

$$\Delta V = f \cdot t = V_0 c_{\beta_0} \mp \left(V^2 - V^2 s_\beta^2\right)^{\frac{1}{2}} = V_0 c_{\beta_0} \mp (\pm) V c_\beta \tag{16}$$

$$\Delta V = V_0 c_{\beta_0} - V c_\beta \tag{17}$$

From Eq. (16), we have $\Delta V - V_0 c_{\beta_0} = \mp(V^2 - V^2 s_\beta^2)^{1/2} = \mp(V^2 - V_0^2 s_{\beta_0}^2)^{1/2}$, which after squaring yields

$$V^2 = V_0^2 + \Delta V^2 - 2\Delta V\ V_0 c_{\beta_0} \tag{18}$$

which conversely represents V as a function of time because $\Delta V = f \cdot t$. The initial yaw angle β_0 must still be determined. In a similar way, Eq. (12) for $\mathrm{d}i/\mathrm{d}V$ can be integrated to provide an expression for the evolution of the inclination in time. Using $V s_\beta = V_0 s_{\beta_0}$ and performing the integration yields

$$\Delta i = -\frac{2}{\pi}\beta_0 + \frac{2}{\pi} \sin^{-1}\left(\frac{V_0 s_{\beta_0}}{V}\right) \tag{19}$$

which is equivalent to $\Delta i = -(2/\pi)(\beta - \beta_0)$. Now, because the inverse-sine function in Eq. (19) is double valued in the interval (0, 2π), it is necessary to be careful about the principal value. Thus, we have

$$\Delta i = (2/\pi)(\beta - \beta_0) \qquad \text{if} \qquad \beta < \pi/2$$

$$\Delta i = 2 - (2/\pi)(\beta + \beta_0) \qquad \text{if} \qquad \beta > \pi/2$$

The number 2 appearing as the first term in the right-hand side of the preceding equation is given in radians and corresponds to 114.6 deg. From $\Delta V = V_0 c_{\beta_0} - V c_\beta$, the condition $\Delta V - V_0 c_{\beta_0} < 0$ is identical to $c_\beta > 0$ or $\beta < \pi/2$ or $\sin^{-1}(V_0 s_{\beta_0}/V) < \pi/2$, and the condition $\Delta V - V_0 c_{\beta_0} > 0$ is identical to $\beta > \pi/2$ or $\sin^{-1}(V_0 s_{\beta_0}/V) > \pi/2$ such that the Edelbaum analysis leads to the following set of equations.

1) If $\Delta V - V_0 c_{\beta_0} < 0$, then

$$\begin{aligned} V &= \left(V_0^2 - 2V_0 \Delta V c_{\beta_0} + \Delta V^2\right)^{\frac{1}{2}} \\ \Delta i &= \frac{2}{\pi} \sin^{-1}\left(\frac{V_0 s_{\beta_0}}{V}\right) - \frac{2}{\pi}\beta_0 = \frac{2}{\pi}(\beta - \beta_0) \end{aligned} \tag{20}$$

2) If $\Delta V - V_0 c_{\beta_0} > 0$, then

$$\begin{aligned} V &= \left(V_0^2 - 2V_0 \Delta V c_{\beta_0} + \Delta V^2\right)^{\frac{1}{2}} \\ \Delta i &= 2 - \frac{2}{\pi} \sin^{-1}\left(\frac{V_0 s_{\beta_0}}{V}\right) - \frac{2}{\pi}\beta_0 = 2 - \frac{2}{\pi}(\beta + \beta_0) \end{aligned} \tag{21}$$

These equations due to Edelbaum show that one must monitor the condition $\Delta V - V_0 c_{\beta_0}$ and use Eq. (20) to describe the transfer starting from time 0 and later switch to Eqs. (21) as soon as $t = \Delta V/f$ exceeds $V_0 c_{\beta_0}/f$, which will take place for large transfers, as will be shown later by an example. This complication is the direct result of the adoption of the velocity as the independent variable. For large transfers, ΔV as given in Eq. (16) could become double valued in V such that one must use $\Delta V = V_0 c_{\beta_0} - (V^2 - V_0^2 s_{\beta_0}^2)^{1/2}$ from V_0 to $V_0 s_{\beta_0}$ and $\Delta V = V_0 c_{\beta_0} + (V^2 - V_0^2 s_{\beta_0}^2)^{1/2}$ from $V = V_0 s_{\beta_0}$ to V_f, where $V_0 s_{\beta_0} < V_f$. This minimum velocity takes place when $\Delta V = V_0 c_{\beta_0}$, indicating that the orbit will grow to become larger than the final desired orbit and later shrink to that desired orbit. This will happen when larger inclination changes are required because then the orbit plane rotation will be carried out mostly at those higher intermediate altitudes. This is the result of the trade between inclination and radius or velocity.

Let us now solve for the initial yaw angle β_0 from the terminal conditions at time t_f. At $t = t_f$, $V = V_f$ and $\Delta i = \Delta i_f$. Let us also provide a uniformly valid expression for the yaw angle β, which will allow us to replace Edelbaum's two distinct Δi expressions in Eqs. (20) and (21) by a single expression uniformly valid for all transfers. Using Eq. (20) for Δi, we get $(\pi/2)\Delta i_f + \beta_0 = \sin^{-1}(V_0 s_{\beta_0}/V_f)$, which after expansion and division by c_{β_0} yields

$$\tan\beta_0 = \frac{\sin[(\pi/2)\Delta i_f]}{(V_0/V_f) - \cos[(\pi/2)\Delta i_f]} \tag{22}$$

Now carrying out the same manipulations using Eq. (21), we get $(\pi/2)[\Delta i_f - 2 + (2/\pi)\beta_0] = -\sin^{-1}(V_0 s_{\beta_0}/V_f)$, which after expansion and division by c_{β_0} yields the same expression as in Eq. (22), which is then valid regardless of whether $\Delta V - V_0 c_{\beta_0} < 0$ or $\Delta V - V_0 c_{\beta_0} > 0$. Now from $\Delta V = V_0 c_{\beta_0} - V c_\beta$, and using Eq. (18) for the velocity, the yaw angle β at future times is given by the following expression without ambiguity:

$$\beta = \cos^{-1}\left[\frac{V_0 c_{\beta_0} - \Delta V}{\left(V_0^2 - 2V_0 \Delta V c_{\beta_0} + \Delta V^2\right)^{\frac{1}{2}}}\right] \tag{23}$$

where $\Delta V = f \cdot t$ and $0 \le \beta \le \pi$. Edelbaum's expression $V s_\beta = V_0 s_{\beta_0}$ would yield $\beta = \sin^{-1}(V_0 s_{\beta_0}/V) \le \pi/2$, even though β could, for large transfers, exceed $\pi/2$. If the evolution of Δi as a function of time or velocity or ΔV is desired, Eq. (20) for Δi is used until $\Delta V = V_0 c_{\beta_0}$. When $\Delta V - V_0 c_{\beta_0} > 0$, Δi as given in Eq. (21) is used. However, in Eq. (21), the inverse-sine function will always return a β angle that is always less than $\pi/2$, and this value for β is the correct value to be used in $\Delta i = 2 - (2/\pi)(\beta + \beta_0)$. This β angle is clearly not the real yaw angle because in this case it would be given by $\pi - \beta$ with $\beta < \pi/2$, such that the yaw angle is now larger than $\pi/2$. If β is adjusted this way, then the correct β profile can be obtained with Edelbaum's theory. However, we can remove this ambiguity if we make the following observation. If the real β angle is used in $\Delta i = 2 - (2/\pi)(\beta + \beta_0)$, we get a further simplification of the Edelbaum results such that

$$\Delta i = 2 - (2/\pi)(\pi - \beta + \beta_0) = (2/\pi)(\beta - \beta_0) \tag{24}$$

Equation (24) is universally valid for all yaw angles $0 \le \beta < \pi$ and should be the only one used. It will effectively replace Edelbaum's

Eqs. (20) and (21) provided that the angle β is computed from Eq. (23). Because the sign of $\Delta V - V_0 c_{\beta_0}$ is effectively accounted for in Eq. (23), it will return the yaw angle β to be used in Eq. (24) for the unambiguous evaluation of Δi. Let us now find expressions for s_{β_0}, c_{β_0}, s_β, and c_β directly by using the identity in Eq. (24), or

$$\cos[(\pi/2)\Delta i] = c_\beta c_{\beta_0} + s_\beta s_{\beta_0} \tag{25}$$

$$\sin[(\pi/2)\Delta i] = s_\beta c_{\beta_0} - s_{\beta_0} c_\beta \tag{26}$$

If we multiply Eq. (25) by VV_0 and replace Vs_β by $V_0 s_{\beta_0}$ and Vc_β by $V_0 c_{\beta_0} - \Delta V$, we get, after regrouping terms,

$$c_{\beta_0} = \frac{V_0 - V\cos[(\pi/2)\Delta i]}{\Delta V} \tag{27}$$

In a similar manner, if we replace this time $V_0 s_{\beta_0}$ by Vs_β and $V_0 c_{\beta_0}$ by $\Delta V + Vc_\beta$, we get an expression for c_β:

$$c_\beta = \frac{V_0\cos[(\pi/2)\Delta i] - V}{\Delta V} \tag{28}$$

Equation (26) can also be written as $VV_0\sin[(\pi/2)\Delta i] = Vs_\beta V_0 c_{\beta_0} - V_0 s_{\beta_0} V c_\beta$. If we now replace Vs_β by $V_0 s_{\beta_0}$ and $V_0 c_{\beta_0}$ by $\Delta V + Vc_\beta$, then we obtain an expression for s_{β_0}:

$$s_{\beta_0} = \frac{V\sin[(\pi/2)\Delta i]}{\Delta V} \tag{29}$$

If, on the other hand, we replace $V_0 c_{\beta_0}$ by $\Delta V + Vc_\beta$, then the identity will yield the expression for s_β:

$$s_\beta = \frac{V_0\sin[(\pi/2)\Delta i]}{\Delta V} \tag{30}$$

Now these expressions will readily yield β_0 and β by direct use of the ATAN2 routine because

$$\tan\beta_0 = \frac{V\sin[(\pi/2)\Delta i]}{V_0 - V\cos[(\pi/2)\Delta i]} = \frac{s_{\beta_0}}{c_{\beta_0}} \tag{31}$$

$$\tan\beta = \frac{V_0\sin[(\pi/2)\Delta i]}{V_0\cos[(\pi/2)\Delta i] - V} = \frac{s_\beta}{c_\beta} \tag{32}$$

The last two expressions can be used to obtain the initial β_0 and current β provided that the appropriate Δi expression of Eqs. (20) or (21) is used according to whether $\Delta V - V_0 c_{\beta_0}$ is positive or negative. Although Eqs. (31) and (32) are valid for any optimal (V, Δi) pair during the transfer, there is clearly a singularity at time 0 when $V = V_0$ and $\Delta i = 0$. The angle β_0 is best obtained by setting $V = V_f$ and $\Delta i = \Delta i_f$ in Eq. (31). It is better to use Eq. (23) for the control time history instead of Eq. (32) because we do not have to switch between two Δi expressions to describe that evolution in the first case. Finally, Edelbaum's ΔV equation in terms of the velocities and inclination is obtained from the velocity equation $V = (V_0^2 - 2V_0\Delta V c_{\beta_0} + \Delta V^2)^{1/2}$. If we square this expression, replace ΔV in the product term by $V_0 c_{\beta_0} - Vc_\beta$, and then use the identity $c_\beta c_{\beta_0} = \frac{1}{2}c_{\beta-\beta_0} + \frac{1}{2}c_{\beta+\beta_0}$, with $c_{\beta-\beta_0} = \cos[(\pi/2)\Delta i]$ from Eq. (24), then we get, with $Vs_\beta = V_0 s_{\beta_0}$, $V^2 = -V_0^2 + V_0^2 s_{\beta_0}^2 + VV_0\cos[(\pi/2)\Delta i] + V_0 c_{\beta_0} V c_\beta + \Delta V^2$. However, $V_0^2 s_{\beta_0}^2 = V_0 s_{\beta_0} V s_\beta$ and, if this term is combined with $V_0 c_{\beta_0} V c_\beta$, the result will be $VV_0 c_{\beta-\beta_0}$, which can be replaced by $VV_0\cos[(\pi/2)\Delta i]$. The final result is given by $V^2 = -V_0^2 + 2VV_0\cos[(\pi/2)\Delta i] + \Delta V^2$, from which

$$\Delta V = \left\{V_0^2 - 2VV_0\cos[(\pi/2)\Delta i] + V_2\right\}^{\frac{1}{2}} \tag{33}$$

This is Edelbaum's ΔV equation for constant-acceleration circle-to-inclined-circle transfer. It is valid for any (V, Δi) pair along the transfer, provided, once again, that the appropriate Δi expression is used, i.e., Eq. (20) or (21) according to whether $\Delta V - V_0 c_{\beta_0}$ is < 0 or > 0, respectively. As shown earlier, ΔV is double valued in the velocity because Δi itself is double valued in that same variable. However, Eq. (33) is mainly used to obtain the total ΔV_{tot} required to achieve a given transfer between V_0 and V_f with a relative inclination change of Δi_f. It is valid for any $0 < \Delta i_f < 114.6$ deg ($0 < \Delta i_f < 2$ rad) because this is the limiting Δi in Eq. (21). The transfer time t_f is simply obtained from $t_f = \Delta V_{\text{tot}}/f$.

III. Formulation Using Optimal Control Theory

Let the system equations be given by Eqs. (10) and (11) with time as the independent variable and i and V as the state variables. The yaw angle β is the control variable. This problem is now cast as a minimum time transfer problem between initial and final parameters i_0, V_0 and i_f, V_f, respectively. The variational Hamiltonian is then given by

$$H = 1 + \lambda_i[(2/\pi)(f/V)s_\beta] + \lambda_V(-fc_\beta) \tag{34}$$

because the performance index is simply given by

$$J = \int_{t_0}^{t_f} L\,\mathrm{d}t$$

with $L = 1$. The Euler–Lagrange differential equations are given by

$$\dot{\lambda}_V = -\frac{\partial H}{\partial V} = \frac{2}{\pi}\frac{fs_\beta}{V^2}\lambda_i \tag{35}$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial i} = 0 \tag{36}$$

Therefore, λ_i is a constant. The optimality condition is given by

$$\frac{\partial H}{\partial \beta} = \lambda_i\frac{2}{\pi}\frac{f}{V}c_\beta + f\lambda_V s_\beta = 0 \tag{37}$$

which yields the optimal control law

$$\tan\beta = -(2/\pi)(\lambda_i/V\lambda_V) \tag{38}$$

There is no need to integrate $\dot{\lambda}_V$ because we can use the transversality condition $H_f = 0$ at the final time. The Hamiltonian is a constant of the motion because it is not an explicit function of time. Therefore, it is equal to zero all of the time. Therefore, we can solve for λ_V and λ_i from Eq. (37) and

$$H = 0 = 1 + (2/\pi)(f/V)s_\beta\lambda_i - fc_\beta\lambda_V$$

This results in

$$\lambda_i = -(\pi s_\beta V/2f) = \text{const} \tag{39}$$

$$\lambda_V = c_\beta/f \tag{40}$$

The so-called convexity condition or strengthened Legendre–Clebsch condition in the calculus of variations[9,10] is also satisfied by this local minimum because

$$\frac{\partial^2 H}{\partial \beta^2} = -\lambda_i\frac{2}{\pi}\frac{f}{V}s_\beta + f\lambda_V c_\beta = s_\beta^2 + c_\beta^2 = 1$$

by replacement of λ_i and λ_V from Eqs. (39) and (40). Therefore, $\partial^2 H/\partial\beta^2 > 0$ is strictly positive. Equation (39) reveals that $Vs_\beta = V_0 s_{\beta_0}$ because the acceleration f is assumed to be a constant. We can now take advantage of this constancy of Vs_β to integrate the velocity equation (11). This yields, as in the preceding section, an expression for V as a function of time, $V = (V_0^2 + f^2t^2 - 2f\cdot tV_0 c_{\beta_0})^{1/2}$. We can also obtain the preceding equation without integrating $\mathrm{d}V/\mathrm{d}t$ by simply writing

$$V = \frac{V_0 s_{\beta_0}}{s_\beta} = V_0 s_{\beta_0}\frac{(1+\tan^2\beta)}{\tan\beta}$$

However, an expression for $\tan\beta$ is needed first. From Eq. (38),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tan\beta) = \frac{2}{\pi}\lambda_i\frac{(\dot{V}\lambda_V + V\dot{\lambda}_V)}{V^2\lambda_V^2}$$

Replacing $\dot{V}$ and $\dot{\lambda}_V$ by Eqs. (11) and (35) and using Eqs. (39) and (40) to eliminate λ_i and λ_V, the preceding derivative can be written as $(\mathrm{d}/\mathrm{d}\beta)(\tan\beta)\dot{\beta} = \dot{\beta}/c_\beta^2 = fs_\beta/Vc_\beta^2$, which yields $\dot{\beta} = fs_\beta/V$. Because $Vs_\beta = V_0 s_{\beta_0}$, this can also be written as $\dot{\beta} = fs_\beta^2/V_0 s_{\beta_0}$, which, after integration, yields the control law

$$\tan\beta = \frac{V_0 s_{\beta_0}}{V_0 c_{\beta_0} - f\cdot t} \tag{41}$$

Now the inclination time history can be obtained by direct integration of Eq. (10) between the limits 0 and i by using the expression for V,

$$\Delta i = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{f\cdot t - V_0 c_{\beta_0}}{V_0 s_{\beta_0}}\right) + \frac{\pi}{2} - \beta_0\right] \tag{42}$$

This formula is uniformly valid for all t, unlike the formulation of the preceding section, which resulted in a set of two expressions for Δi because Δi was double valued in the velocity. This simplification is achieved because time is selected as the independent variable instead of the velocity. If we integrate Eq. (35) for $\dot\lambda_V$ and use Eq. (39) to eliminate the constant λ_i, then

$$\dot\lambda_V = -\left(V_0^2 s_{\beta_0}^2/V^3\right) = -V_0^2 s_{\beta_0}^2\left(V_0^2 + f^2t^2 - 2V_0 c_{\beta_0} f\cdot t\right)^{-\frac{3}{2}}$$

which upon integration yields

$$\lambda_V = \frac{V_0 c_{\beta_0} - f\cdot t}{fV}$$

with $(\lambda_V)_0 = c_{\beta_0}/f$ and, in view of Eq. (40),

$$c_\beta = \frac{V_0 c_{\beta_0} - f\cdot t}{V}$$

From the definition of the influence functions λ_i and λ_V, we have $(\lambda_V)_0 = c_{\beta_0}/f = \partial t_f/\partial V_0$, such that the sensitivity of the total transfer time t_f to small variations in the initial velocity V_0 will be given by

$$\delta t_f = \left(c_{\beta_0}\big/ f\right)\delta V_0$$

Similarly from

$$(\lambda_i)_0 = -\frac{\pi V_0 s_{\beta_0}}{2f} = \frac{\partial t_f}{\partial(\Delta i)_0}$$

we have, for the sensitivity of the total transfer time t_f to small variations in the initial inclination,

$$\delta t_f = -\frac{\pi V_0 s_{\beta_0}}{2f}\delta(\Delta i)_0$$

Let us now obtain an alternate expression for ΔV in terms of β_0, V_0, and Δi. From Eq. (42), we have

$$\frac{f\cdot t - V_0 c_{\beta_0}}{V_0 s_{\beta_0}} = \tan\left(\frac{\pi}{2}\Delta i + \beta_0 - \frac{\pi}{2}\right) = \frac{-1}{\tan[(\pi/2)\Delta i + \beta_0]}$$

and, because $f\cdot t = \Delta V$,

$$\Delta V = V_0 c_{\beta_0} - \frac{V_0 s_{\beta_0}}{\tan[(\pi/2)\Delta i + \beta_0]} \tag{43}$$

This expression can be used to evaluate the total ΔV for the desired transfer once β_0, the initial yaw angle, is known. To obtain β_0, let us first observe that, during the integration of the $\dot V$ equation (11), the same intermediate results shown in the preceding section and resulting from that particular integration are valid here too. They are $\Delta V = f\cdot t = V_0 c_{\beta_0} - V c_\beta$, such that

$$\Delta V = V_0 c_{\beta_0} \pm \left(V^2 - V_0^2 s_{\beta_0}^2\right)^{\frac{1}{2}}$$

From the Δi equation (42) and using the control law for $\tan\beta$ given in Eq. (41), we get

$$\Delta i = (2/\pi)(\beta - \beta_0)$$

We can now obtain β_0 by using the expression for ΔV just given and writing Δi as

$$\Delta i = \frac{2}{\pi}\left\{\tan^{-1}\left[\frac{\pm\left(V^2 - V_0^2 s_{\beta_0}^2\right)^{\frac{1}{2}}}{V_0 s_{\beta_0}}\right] + \frac{\pi}{2} - \beta_0\right\}$$

Because $\tan^{-1} x = \pm\cos^{-1}[1/(x^2+1)^{1/2}]$ according to whether x is >0 or <0, the preceding expression for Δi can be cast as

$$\Delta i = \frac{2}{\pi}\left[\pm\cos^{-1}\left(\frac{V_0 s_{\beta_0}}{V}\right) + \frac{\pi}{2} - \beta_0\right]$$

which after expansion and division by c_{β_0} yields the same result as Eq. (31). As in Eq. (31), β_0 is obtained by way of the ATAN2 routine because the numerator and denominator are equivalent to s_{β_0} and c_{β_0}, respectively, as was shown in the preceding section:

$$\tan\beta_0 = \frac{\sin[(\pi/2)\Delta i]}{(V_0/V) - \cos[(\pi/2)\Delta i]} \tag{44}$$

From $\Delta V = V_0 c_{\beta_0} - V c_\beta$ and using the expressions for c_{β_0} and c_β developed in the preceding section, namely, Eqs. (27) and (28), Edelbaum's ΔV equation (33) is readily obtained. For given V_0, V_f, and $(\Delta i)_f$, β_0 can be obtained from the preceding equation, which then allows us to describe V and Δi as well as β as a function of time from $V = (V_0^2 + f^2t^2 - 2f\cdot tV_0 c_{\beta_0})^{1/2}$ and Eqs. (42) and (41), respectively. Equation (44) shows that, as Δi approaches the 2 rad value or $\Delta i = 114.59$ deg, $\sin[(\pi/2)\Delta i]$ will approach zero so that $\beta_0 \to 0$, indicating that the initial phase of the transfer will be coplanar. The ΔV equation of Edelbaum given by Eq. (33) will then approach $\Delta V = V_0 + V_f$, which is the sum of the initial and final velocities. Because V_0 represents the ΔV needed to transfer from V_0 to ∞ or escape and because V_f represents the ΔV needed to transfer from ∞ back to V_f, then the transfer is initially coplanar until escape. At infinity, $V_\infty = 0$ and the inclination change is achieved at zero cost, after which the return leg to V_f is also coplanar, resulting in $\Delta V = V_0 + V_f$. This is shown from $\Delta V_1 = (V_0^2 - 2V_0V_\infty + V_\infty^2)^{1/2} = V_0$ and $\Delta V_2 = (V_\infty^2 - 2V_\infty V_f + V_f^2)^{1/2} = V_f$, such that the transfer requires a total ΔV of $\Delta V = \Delta V_1 + \Delta V_2 = V_0 + V_f$. The transfer time t_f can no longer be obtained from $t_f = \Delta V/f$ because it will approach infinity. In this case, it is not economical to use continuous thrust to achieve escape with cost V_0 because we can achieve parabolic escape by intermittent thrusting at perigee such that the cumulative ΔV from these infinitesimal firings will be equal to the impulsive escape ΔV of $[\sqrt{2} - 1]V_0$, which is only about 41% of the continuous velocity change V_0. The same argument holds true for the return leg after carrying out the orbit rotation for free at infinity. The transfer then mimics the well-known impulsive biparabolic mode. In practice, for the continuous thrust case, a high intermediate orbit is achieved to carry out the rotation and keep the transfer time finite. For any inclination larger than $\Delta i = 114.59$ deg, the cost of the orbit rotation is zero and the ΔV remains stationary at $V_0 + V_f$. If the transfer is purely coplanar, i.e., $\Delta i = 0$, then $\Delta V = |V_0 - V_f|$, the difference of the boundary velocities. For given V_0 and V_f, ΔV reaches a maximum if we set $\partial\Delta V/\partial\Delta i = 0$, where $\Delta V = \{V_0^2 - 2V_fV_0\cos[(\pi/2)\Delta i] + V_f^2\}^{1/2}$. This results in $\sin[(\pi/2)\Delta i] = 0$ or $\Delta i = 114.59$ deg as discussed earlier. Therefore, Edelbaum's Eq. (33) is to be used for $0 \le \Delta i \le 114.59$ deg only. For $\Delta i > 114.59$ deg, $\Delta V = V_0 + V_f$, as the use of Eq. (33) in this case will yield the wrong ΔV. Furthermore, from $\Delta V = V_0 c_{\beta_0} \pm (V^2 - V_0^2 s_{\beta_0}^2)^{1/2}$, as Δi approaches 114.59 deg from below, ΔV must approach $V_0 + V_f$, which implies that $\beta_0 \to 0$, as shown earlier, and $\beta_f \to 180$ deg. The ΔV in Eq. (43) also approaches $V_0 + V_f$ because with $\beta_0 = \varepsilon$, $s_{\beta_0} \sim \beta_0$, $c_{\beta_0} \sim 1$, $s_{\beta_f} \sim -s_{\beta_0} \sim -\beta_0$, and $(\pi/2)\Delta i \sim 180$ deg. Therefore,

$$\Delta V \cong V_0 - \frac{V_f s_{\beta_f}}{\tan(\pi + \varepsilon)} \cong V_0 + \frac{V_f\varepsilon}{\varepsilon} = V_0 + V_f$$

IV. Results and Numerical Comparison

In summary, we have the following algorithm. First, one computes V_0 and V_f from the knowledge of initial and final semimajor axes a_0 and a_f, $V_0 = (\mu/a_0)^{1/2}$ and $V_f = (\mu/a_f)^{1/2}$. Given $\mu = 398{,}601.3$ km^3/s^2 (Earth's gravity constant), Δi the total inclination change desired, and f the low-thrust acceleration, one computes β_0 from Eq. (44) and the total ΔV_{tot} from Eq. (43) such that the transfer time is known from $t_f = \Delta V_{\text{tot}}/f$. The variation with time of the various variables of interest is obtained from

$$\Delta V = f\cdot t \qquad \beta = \tan^{-1}\left(\frac{V_0 s_{\beta_0}}{V_0 c_{\beta_0} - f\cdot t}\right)$$

$$V = \left(V_0^2 - 2V_0 f\cdot t\, c_{\beta_0} + f^2t^2\right)^{\frac{1}{2}} \qquad \lambda_V = c_\beta/f$$

$$\Delta i = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{f\cdot t - V_0 c_{\beta_0}}{V_0 s_{\beta_0}}\right) + \frac{\pi}{2} - \beta_0\right]$$

where λ_i is, of course, constant and given by $-\pi V_0 s_{\beta_0}/(2f)$. The total $(\Delta i)_f$ is obtained from $|i_0 - i_f|$, where i_0 and i_f are the initial and final inclination, respectively. If $i_f > i_0$, the current inclination i is given by $i = i_0 + \Delta i$. If $i_f < i_0$, then $i = i_0 - \Delta i$. This is needed because we assumed $\Delta i > 0$ so that $\beta > \beta_0$, too.

We now show the applicability of this analysis by considering three distinct examples of orbit transfer from the same initial semimajor axis $a_0 = 7000$ km and increasing inclinations, to the same final geostationary orbit of $a_f = 42{,}166$ km, $i_f = 0$ deg. In the first example with $i_0 = 28.5$ deg, β is always less than $\pi/2$, resulting in a uniform expansion of the transfer orbit. For the large i_0 example with $i_0 = 90$ deg, β will cover the range beyond $\pi/2$ with the transfer orbit expanding beyond the target orbit and later shrinking to match it. Finally, for the limiting $i_0 = 114.591$ deg, β switches from 0 to π at infinity, where the orbit rotation is carried out for free. These three representative examples are now shown in more detail. An example of a constant-acceleration, low-Earth-orbit (LEO)-to-geostationary-Earth-orbit (GEO) transfer is shown in Figs. 1 and 2. The transfer is from $a_0 = 7000$ km, $i_0 = 28.5$ deg to $a_f = 42{,}166$ km, $i_f = 0.0$ deg with $f = 3.5 \times 10^{-7}$ km/s^2. The total transfer time $t_f = 191.26259$ days with corresponding $\Delta V_{\text{tot}} = 5.78378$ km/s. The thrust yaw angle increases from its initial value $\beta_0 = 21.98$ deg to $\beta_f = 66.75$ deg, as it is more efficient to rotate the orbit plane at higher altitudes.

Figures 3 and 4 show the variation of the thrust yaw angle β, semimajor axis a, velocity V, and inclination i as a function of time for a low-thrust transfer between $a_0 = 7000$ km, $i_0 = 90.0$ deg and $a_f = 42{,}166$ km, $i_f = 0.0$ deg orbit. The acceleration is still constant at $f = 3.5 \times 10^{-7}$ km/s^2. Starting from its initial value at $\beta_0 = 10.92$ deg, β stays almost stationary for the first 100 days before surging to the final value $\beta_f = 152.29$ deg. It goes through 90 deg at day 245, when it starts to decelerate the vehicle because the orbit radius has exceeded the desired final altitude and thus must be shrunk until the final transfer time of $t_f = 335$ days for a total $\Delta V_{\text{tot}} = 10.13$ km/s. This is shown in Fig. 4, where the intermediate velocity is much less than the final desired velocity of 3.07 km/s at GEO.

Figures 5 and 6 are for $i_0 = 114.591$ deg, the uppermost limit of Edelbaum's theory. The β angle stays at zero, indicating a coplanar transfer up to $t = 250$ days, where $V \cong 0$ and a is infinity, before flipping to $\beta = 180$ deg for the return leg to GEO. The inclination change is carried out instantaneously at infinity with zero cost. The total transfer time is $t_f = 351$ days with a ΔV of 10.61 km/s. This algorithm is valid regardless of whether the transfer is to a higher orbit or a lower orbit, irrespective of the direction of the inclination change. Finally, it is important to compare Edelbaum's ΔV to the exact ΔV generated by a dynamic model that uses the exact six-state formulation. The averaged form of this complete six-state dynamics

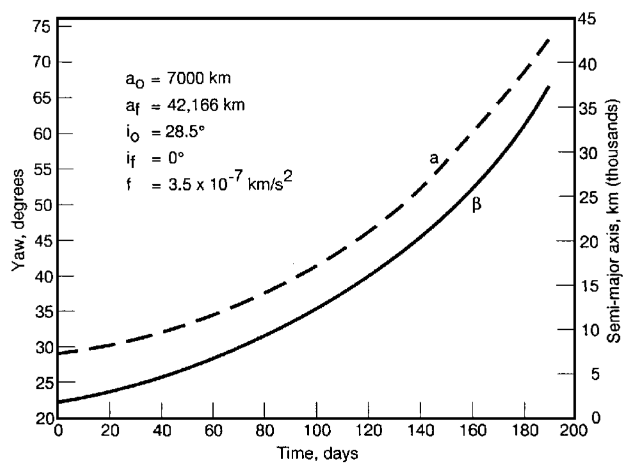

Fig. 1 Optimal thrust yaw and semimajor axis for a low-acceleration LEO-to-GEO transfer.

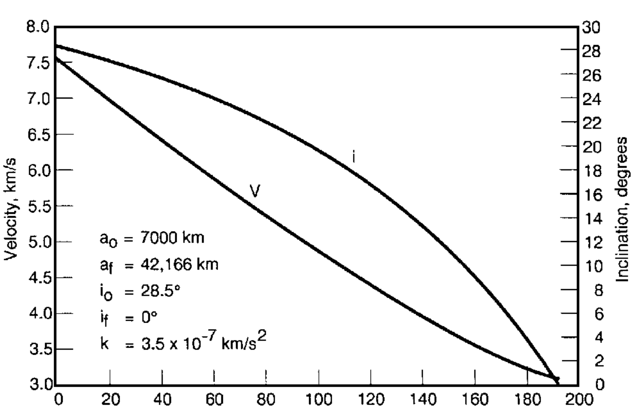

Fig. 2 Optimal velocity and inclination for a low-acceleration LEO-to-GEO transfer.

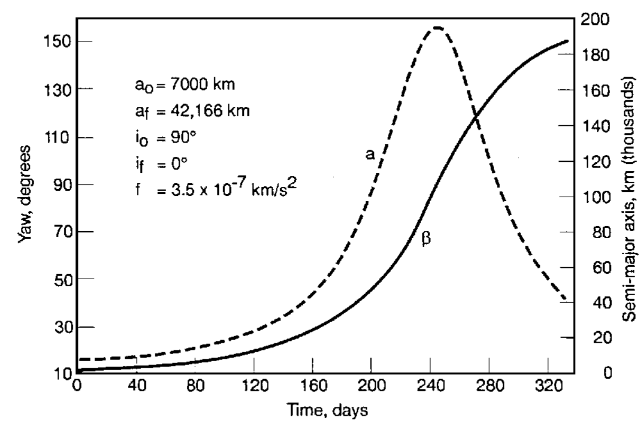

Fig. 3 Optimal thrust yaw and semimajor axis for a large inclination change transfer.

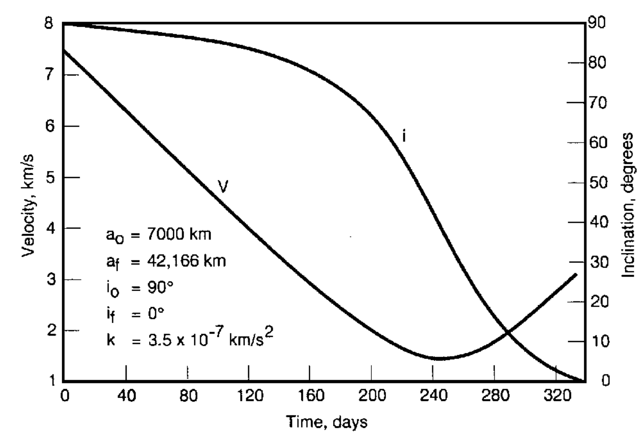

Fig. 4 Optimal velocity and inclination for a large inclination change transfer.

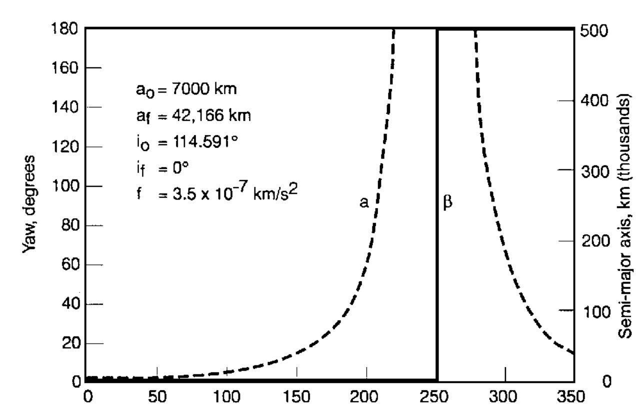

Fig. 5 Optimal thrust yaw and semimajor axis for limiting case of Edelbaum's theory.

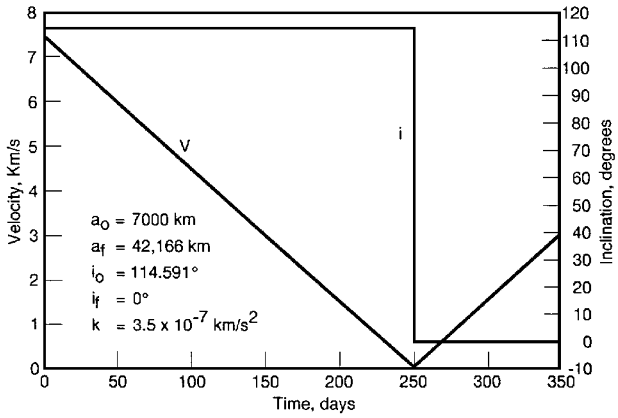

Fig. 6 Optimal velocity and inclination for limiting case of Edelbaum's theory.

has been used in Refs. 11 and 12 to find the minimum time transfer solution corresponding to the example of Figs. 1 and 2. It was further assumed that e_0, ω_0, Ω_0 and e_f, ω_f, Ω_f are equal to zero, with M_0 and M_f being irrelevant because the spacecraft position is averaged out. A minimum transfer time of $t_f = 1.610144786 \times 10^7$ s was found, corresponding to a ΔV of 5.635506 km/s using the same acceleration of $f = 3.5 \times 10^{-7}$ km/s^2. This solution is essentially identical to the one obtained by precision integration without averaging, and it is about 148 m/s less than Edelbaum's value. This difference can be attributed to the nonoptimality of the β profile within each cycle in Edelbaum's theory because this angle is held constant within each such cycle, unlike the optimal pitch and yaw profiles used by the more elaborate exact dynamics. Edelbaum's theory does not account for any gravity losses because it assumes that the transfer orbit is always circular, unlike the optimal solution, which uses a nonzero thrust pitch profile and a variable eccentricity orbit during the transfer. However, this is more than compensated by the losses incurred from using constant yaw within each cycle by Edelbaum's theory; hence, the overall loss is 148 m/s. Although these losses are small, the transfer histories corresponding to the use of Edelbaum's and the exact guidance schemes may be quite different, introducing thereby different radiation degradation profiles during transit in the Van Allen belts, which would have an important impact on the overall system performance.

V. Concluding Remarks

Edelbaum's analytic low-thrust, circle-to-inclined-circle orbit transfer theory has been revisited and later reformulated as a minimum-time transfer problem. An essential simplification is obtained by replacing a set of two expressions for the orbital inclination due to Edelbaum by a single analytic expression uniformly valid for all transfers. This simplification is achieved within both the framework of Edelbaum's original analysis and the new formulation using optimal control theory. Additional expressions for the initial value of the control parameter needed for a given transfer as well as its functional dependency on time are also presented using inverse-tangent functions without any quadrant ambiguity. This analytic theory, as well as its many future extensions, is crucial for spacecraft systems analysis and design optimization applications and future autonomous onboard guidance implementation. This analysis has also considerable academic value due to its complete analytic closure characteristics. A distinct challenge consists of allowing the transfer orbit to be elliptical, which is an inevitable outcome of shadowing and, therefore, intermittent thrusting, within a completely analytic framework.

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